

Counting Spanning Trees on Fractal Graphs

Jason A. Anema

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Abstract

Using the method of spectral decimation and a modified version of Kirchhoff's Matrix-Tree Theorem, a closed form solution to the number of spanning trees on approximating graphs to a fully symmetric self-similar structure on a finitely ramified fractal is given in Theorem (3.4). Examples calculated include the Sierpiński Gasket, a non-p.c.f. analog of the Sierpiński Gasket, the Diamond fractal, and the Hexagasket. For each example, the asymptotic complexity constant is found.

Dropping the fully symmetry assumption, it is shown that the limsup and liminf of the asymptotic complexity constant exist.

1 Introduction

The problem of counting the number of spanning trees in a finite graph dates back more than 150 years. It is one of the oldest and most important graph invariants, and has been actively studied for decades. Kirchhoff's famous Matrix-Tree Theorem [29], appearing in 1847, relates properties of electrical networks and the number spanning trees. There are now a large variety of proofs for the Matrix-Tree Theorem, for some examples see [7, 12, 26]. Counting spanning trees is a problem of fundamental interest in mathematics [6, 49, 11, 30, 8, e.g.] and physics [50, 52, 21, 51, 19, e.g.]. Its relation to probability theory was explored in [31, 33]. It has found applications in theoretical chemistry, relating to the enumeration of certain chemical isomers [10], and as a measure of network reliability in the theory of networks [16].

Recently, S.C. Chang et al. studied the number of spanning trees and the associated asymptotic complexity constants on regular lattices in [14, 15, 38, 48]. These types of problems led them to consider spanning trees on self-similar fractal lattices, as they exhibit scale invariance rather than translation invariance. In [13] S.C. Chang, L.C. Chen, and W.S. Yang calculate the number of spanning trees on the sequence of graph approximations to the Sierpiński Gasket of dimension two, three and four, as well as for two generalized Sierpiński Gaskets ($SG_{2,3}(n)$ and $SG_{2,4}(n)$), and conjecture a formula for the number of spanning trees on the $d - \text{dimensional}$ Sierpiński Gasket at stage n , for general d . Their method of proof uses a decomposition argument to derive multi-dimensional polynomial recursion equations to be solved. Independently, that same year, E. Teufl and S. Wagner [42] give the number of spanning trees on the Sierpiński Gasket of dimension two at stage n , using the same argument. In [43] they expand on this work, constructing graphs by a replacement procedure

yielding a sequence of self-similar graphs (this notion of self-similarity is different than in [28]), which include the Sierpiński graphs. For a variety of enumeration problems, including counting spanning trees, they show that their construction leads to polynomial systems of recurrences and provide methods to solve these recurrences asymptotically. Using the same construction technique in [44], they give, under the assumptions of *strong symmetry* (see [44, section 2.2]) and *connectedness*, a closed form equation for the number of spanning trees [44, Theorem 4.2]. This formulation requires calculating the resistance scaling factor and the tree scaling factor (defined in [44, Theorem 4.1]). In Section 8.3.1 they show that the d -dimensional Sierpiński Gasket at stage n , satisfies their assumptions and prove the conjecture of [13]. *Strong Symmetry* is a condition which must be satisfied on each level of construction, whereas the *full symmetry* condition, that will be assumed in the present work, is only a condition on the first level of construction. The Hexagasket, found in section 5.4, is an example of a graph sequence that is fully symmetric, and not strongly symmetric.

In [2, 3] B. Steinhurst, A. Teplyaev, et al., describe the method of spectral decimation for self-similar fully symmetric finitely ramified fractals, which shows how to explicitly calculate the spectrum of the Laplacian on such fractals, generalizing the ideas of [22, 36]. The central result of the present work, Theorem (3.4), relies on their paper to describe how to calculate, in an analytic fashion, the number of spanning trees of the sequence of graph approximations to such fractals. The idea is that the number of spanning trees on finite, connected, loopless, graph is given by a normalized product of the non-zero eigenvalues of the graph's probabilistic Laplacian. The fractal graphs considered here have the advantage that we can calculate those eigenvalues explicitly, as preiterates of a particular rational function. This enables one to be able to calculate their product explicitly, and hence calculate the number of spanning trees explicitly. Section 2 of this work will set up some notation and describe the class of fractal graphs that will be considered. In Section 3 the main result of this work is presented. Theorem (3.4) enables one to write down a closed formula for the number of spanning trees on the class of fractal graphs considered. A nice Corollary of this is the fact that such formulas remain simple. In section 4, it is shown that if we drop the assumption of full symmetry, then the limsup and liminf of the asymptotic complexity constant exist. This section ends with a few related conjectures. Section 5 begins with a well known example, the Sierpiński Gasket. This is done to show how to use Theorem (3.4). The section continues to calculate the number of spanning trees, and the asymptotic complexity constant, for the graph approximations to a non-p.c.f. analog of the Sierpiński Gasket, the Diamond fractal, and the Hexagasket. The author would like to thank Benjamin Steinhurst, Robert Strichartz, and Alexander Teplyaev for their helpful conversations and comments.

2 Background and Preliminaries

2.1 Graph and Probabilistic Graph Laplacians

Kirchhoff's Matrix-Tree Theorem relates a normalized product of the non-zero eigenvalues of the graph Laplacian to the number of spanning trees of a loop-

less connected graph, since fractal graphs are always connected, and loopless we will make this assumption henceforward. However, using the method of spectral decimation one is only able to find the eigenvalues of the probabilistic graph Laplacian for a specified class of fractal graphs, so a suitable version of Kirchhoff's theorem for probabilistic graph Laplacians must be found. Working in that direction, recall that for any graph $T = (V, E)$ having n labelled vertices v_1, v_2, \dots, v_n , with vertex set V and edge set E . The graph Laplacian G on T is defined by $G = D - A$, where $D = ((d_{ij}))$ is the degree matrix on T with $d_{ij} = 0$ for $i \neq j$ and $d_{ii} = \deg(v_i)$, and $A = ((a_{ij}))$ is the adjacency matrix on T with a_{ij} is the number of copies of $\{v_i, v_j\} \in E$. The probabilistic graph Laplacian of T is defined by $P = D^{-1}G$. Let I be the $n \times n$ identity matrix,

$$\chi(G) = |G - xI| = \sum_{i=0}^n c_i^G x^i,$$

and

$$\chi(P) = |P - xI| = \sum_{i=0}^n c_i^P x^i,$$

be the characteristic polynomials of G and P , respectively. Let $S := \{1, 2, \dots, n-1, n\}$. If $\theta \subseteq S$, then let $\bar{\theta}$ denote the complement of θ in S . For any $n \times n$ matrix C and any $\theta \subseteq S$, let $C(\theta)$ denote the principal submatrix of C formed by deleting all rows and columns not indexed by an element of θ . From [17], we have that for any $m \times m$ diagonal matrix B , and any $m \times m$ matrix C ,

$$|B + C| = \sum_{\theta \subseteq S} |B(\bar{\theta})| \cdot |C(\theta)|,$$

where the summation is over all subsets $S = \{1, \dots, m\}$. Using this observation and expanding term by term it follows that

$$c_{n-i}^G = (-1)^{n-i} \sum_{|\theta|=i} |D(\theta)| \cdot |P(\theta)| \quad (1)$$

and

$$c_{n-i}^P = (-1)^{n-i} \sum_{|\theta|=i} |P(\theta)|. \quad (2)$$

Now, assume that T is connected and loopless, expand these polynomials, compare c_1^G with c_1^P and apply Kirchhoff's Matrix Tree Theorem and you will arrive that the following theorem, originally shown in [35]. This is the version of the Matrix-Tree Theorem that will be used in this work.

Theorem 2.1 (Kirchhoff's Matrix-Tree Theorem for Probabilistic Graph Laplacians). For any connected, loopless graph T with n labelled vertices, the number of spanning trees of T is

$$\tau(T) = \left| \frac{\left(\prod_{j=1}^n d_j \right)}{\left(\sum_{j=1}^n d_j \right)} \left(\prod_{j=1}^{n-1} \lambda_j^P \right) \right|,$$

where $\{\lambda_j^P\}_{j=1}^{n-1}$ are the non-zero eigenvalues of P .

2.2 Fractal Graphs

Let (X, d) be a complete metric space. If $f_i : X \rightarrow X$ is a contraction with respect to the metric d for $i = 1, 2, \dots, m$, then there exist a unique non-empty compact subset K of X that satisfies

$$K = f_1(K) \cup \dots \cup f_m(K).$$

K is called the *self-similar set with respect to* $\{f_1, f_2, \dots, f_m\}$.

If each f_i is injective and for any n and for any two distinct words $\omega, \omega' \in W_n = \{1, \dots, m\}^n$ we have

$$K_\omega \cap K_{\omega'} = F_\omega \cap F_{\omega'}$$

where $f_\omega = f_{\omega_1} \circ \dots \circ f_{\omega_n}$, $K_\omega = f_\omega(K)$, F_0 is the set of fixed points of $\{f_1, f_2, \dots, f_m\}$, and $F_\omega = f_\omega(F_0)$, is called a *finitely ramified self-similar set with respect to* $\{f_1, f_2, \dots, f_m\}$

For any self-similar set, K , with respect to $\{f_1, f_2, \dots, f_m\}$. There is a natural sequence of *approximating graphs* V_n with vertex set F_n defined as follows. For all $n \geq 0$ and for all $\omega \in W_n$ define V_0 as the complete graph with vertices F_0 ,

$$F_n := \bigcup_{\omega \in W_n} F_\omega,$$

$$F_\omega := \bigcup_{x \in V_0} F_\omega(x),$$

where $F_\omega := f_{a_n} \circ f_{a_{n-1}} \circ \dots \circ f_{a_1} \circ \omega$ and $\omega = a_1 a_2 \dots a_n$. Also, $x, y \in F_n$ are connected by an edge in V_n if $f_i^{-1}(x)$ and $f_i^{-1}(y)$ are connected by an edge in V_{n-1} for some $1 \leq i \leq m$.

Let K be a compact metrizable topological space and S be a finite set. Also, let F_i be a continuous injection from K to itself $\forall i \in S$. Then, $(K, S, \{F_i\}_{i \in S})$ is called a *self-similar structure* if there exists a continuous surjection $\pi : \Sigma \rightarrow K$ such that $F_i \circ \pi = \pi \circ \sigma_i \forall i \in S$, where $\Sigma = S^{\mathbb{N}}$ the one-sided infinite sequences of symbols in S and $\sigma_i : \Sigma \rightarrow \Sigma$ is defined by $\sigma_i(\omega_1 \omega_2 \omega_3 \dots) = i \omega_1 \omega_2 \omega_3 \dots$ for each $\omega_1 \omega_2 \omega_3 \dots \in \Sigma$

Clearly, if K is the self-similar set with respect to injective contractions $\{f_1, f_2, \dots, f_m\}$, then $(K, \{1, 2, \dots, m\}, \{f_i\}_{i=1}^m)$ is a self-similar structure.

Notice that two non-isomorphic self-similar structures can have the same finitely ramified self-similar set, however the structures will not have the same sequence of approximating graphs V_n . Also, any two isomorphic self-similar structures whose compact metrizable topological spaces are finitely ramified self-similar sets will have approximating graphs which are isomorphic $\forall n \geq 0$.

A *fully symmetric* finitely ramified self-similar structure with respect to $\{f_1, f_2, \dots, f_m\}$ is a self-similar structure $(K, \{1, 2, \dots, m\}, \{f_1, f_2, \dots, f_m\})$ such that K is a finitely ramified self-similar set, and, as in [2], for any permutation $\sigma : F_0 \rightarrow F_0$ there is an isometry $g_\sigma : K \rightarrow K$ that maps any $x \in F_0$ into $\sigma(x)$ and preserves the self-similar structure of K . This means that there is a map $\tilde{g}_\sigma : W_1 \rightarrow W_1$ such that $f_i \circ g_\sigma = g_\sigma \circ f_{\tilde{g}_\sigma(i)} \forall i \in W_1$. The group of isometries g_σ is denoted \mathfrak{G} .

As in [27], the definition of a fully symmetric finitely ramified self-similar structure may be combined into one compact definition.

Definition 2.2. A fractal K is a fully symmetric finitely ramified self-similar set if K is a compact connected metric space with injective contraction maps on a complete metric space $\{f_i\}_{i=1}^m$ such that

$$K = f_1(K) \cup \dots \cup f_m(K).$$

and the following three conditions hold:

1. there exist a finite subset F_0 of K such that

$$f_j(K) \cap f_k(K) = f_j(F_0) \cap f_k(F_0)$$

for $j \neq k$ (this intersection may be empty);

2. if $v_0 \in F_0 \cap f_j(K)$ then v_0 is the fixed point of f_j ;
3. there is a group \mathcal{G} of isometries of K that has a doubly transitive action on F_0 and is compatible with the self-similar structure $\{f_i\}_{i=1}^m$, which means that for any j and any $g \in \mathcal{G}$ there exist a k such that

$$g^{-1} \circ f_j \circ g = f_k.$$

3 Counting Spanning Trees on Fractal Graphs

Let K be a fully symmetric finitely ramified self-similar structure, V_n be its sequence of approximating graphs, and P_n denote the probabilistic graph Laplacian of V_n .

The next two Propositions describe the spectral decimation process, which inductively gives the spectrum of P_n .

The V_0 network is the complete graph on the boundary set and we set $m = |V_0|$. Write P_1 in block form

$$P_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is a square block matrix associated to the boundary points. Since the V_1 network never has an edge joining two boundary points A is the $m \times m$ identity matrix. The Schur Complement of P_1 is

$$S(z) = (A - zI) - B(D - z)^{-1}C$$

Proposition 3.1. (Bajorin, et al.,[2]) For a given fully symmetric finitely ramified self-similar structure K there are unique scalar valued rational functions $\phi(z)$ and $R(z)$ such that for $z \notin \sigma(D)$

$$S(z) = \phi(z)(P_0 - R(z))$$

Now P_0 has entries $a_{ii} = 1$ and $a_{ij} = \frac{-1}{m-1}$ for $i \neq j$. Looking at specific entries of this matrix valued equation we get two scalar valued equations

$$\phi(z) = -(m-1)S_{1,2}(z)$$

and

$$R(z) = 1 - \frac{S_{1,1}}{\phi(z)}.$$

Where $S_{i,j}$ is the i, j entry of the matrix $S(z)$.

Now, we let

$$E(P_0, P_1) := \sigma(D) \bigcup \{z : \phi(z) = 0\}$$

and call $E(P_0, P_1)$ the exceptional set.

Let $mult_D(z)$ be the multiplicity of z as an eigenvalue of D , $mult_n(z)$ be the multiplicity of z as an eigenvalue of P_n , $mult_n(z) = 0$ if and only if z is not an eigenvalue of P_n , and similarly $mult_D(z) = 0$ if and only if z is not an eigenvalue of D . Then we may inductively find the spectrum of P_n with the following Proposition.

Proposition 3.2. (Bajorin, et al.,[2]) For a given fully symmetric finitely ramified self-similar structure K , and $R(z)$, $\phi(z)$, $E(P_0, P_1)$ as above, the spectrum of P_n may be calculate inductively using the following criteria:

1. if $z \notin E(P_0, P_1)$, then

$$mult_n(z) = mult_{n-1}(R(z))$$

2. if $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a removable singularity at z then,

$$mult_n(z) = |V_{n-1}|$$

3. if $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at z , $R(z)$ has a removable singularity at z , and $\frac{\partial}{\partial z}R(z) \neq 0$, then

$$mult_n(z) = m^{n-1}mult_D(z) - |V_{n-1}| + mult_{n-1}(R(z))$$

4. if $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z , and $\phi(z) \neq 0$, then

$$mult_n(z) = m^{n-1}mult_D(z) + mult_{n-1}(R(z))$$

5. if $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z , and $\phi(z) = 0$, then

$$mult_n(z) = m^{n-1}mult_D(z) + |V_{n-1}| + mult_{n-1}(R(z))$$

6. if $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at z , $R(z)$ has a removable singularity at z , and $\frac{\partial}{\partial z}R(z) = 0$, then

$$mult_n(z) = m^{n-1}mult_D(z) - |V_{n-1}| + 2mult_{n-1}(R(z))$$

7. if $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a pole at z , then $mult_n(z) = 0$.

8. if $z \in \sigma(D)$, but $\phi(z)$ and $\phi(z)R(z)$ do not have poles at z , $\phi(z) = 0$ and $R(z)$ has a pole at z , then

$$mult_n(z) = m^{n-1}mult_D(z).$$

After carrying out the inductive calculations using items (1)-(8), define

$$A := \{\alpha : \alpha \text{ satisfies item (2) or (8)}\}$$

for $\alpha \in A$, $\alpha_n := \text{mult}_n(\alpha)$

$$B := \{\beta : \text{for some } n \geq 1, \text{mult}_n(\beta) \neq 0 \text{ and } \text{mult}_{n-1}(R(\beta)) \neq 0\}$$

and for $\beta \in B$, $\beta_n^k := \text{mult}_n(R_{(-k)}(\beta))$.

Since V_n is connected $\text{mult}_n(0) = 1$ for all $n \geq 0$. Again from [2], we get that

$$\sigma(P_n) \setminus \{0\} = \bigcup_{\alpha \in A} \{\alpha\} \bigcup_{\beta \in B} \left[\bigcup_{k=0}^n \{R_{-k}(\beta) : \beta_n^k \neq 0\} \right].$$

Hence the non-zero eigenvalues of P_n are the zeros of polynomials or preiterates of rational functions. To be able to use Theorem 2.1, we need to know how to take the product of preiterates of rational functions of a particular form. The proof of Theorem 3.4 will show that $R(z)$ satisfies the assumptions of the next Lemma, and use this information to be able to calculate the number of spanning trees on the fractal graphs under consideration.

Lemma 3.3. Let $R(z)$ be a rational function such that $R(0) = 0$, $\deg(R(z)) = d$, $R(z) = \frac{P(z)}{Q(z)}$, with $\deg(P(z)) > \deg(Q(z))$. Let P_d be the leading coefficient of $P(z)$. Fix $\alpha \in \mathbb{C}$. Let $\{R_{(-n)}(\alpha)\}$ be the set of n^{th} preiterates of α under $R(z)$. By convention, $R_{(0)}(\alpha) := \{\alpha\}$. Then for $n \geq 0$,

$$\prod_{z \in \{R_{(-n)}(\alpha)\}} z = \alpha \left(\frac{-Q(0)}{P_d} \right)^{\left(\frac{d^n - 1}{d - 1} \right)}.$$

Proof of Lemma 3.3. For $n = 0$, the result is clear. For $n = 1$, we note

$$\begin{aligned} \{R_{(-1)}(\alpha)\} &= \{z : R(z) = \alpha\} \\ &= \{z : P(z) - \alpha Q(z) = 0\} \\ &= \{z : P_d z^d + \dots - Q(0)\alpha = 0\}, \end{aligned}$$

where $Q(0)$ is the constant term of $Q(z)$. As the product of the roots of a polynomial is equal to the constant term over the coefficient of the highest degree term, we have that

$$\prod_{z \in \{R_{(-1)}(\alpha)\}} z = \frac{-\alpha Q(0)}{P_d}.$$

Assume our equation holds for n . Then for $n + 1$ we have

$$\{w : w \in R_{(-(n+1))}(\alpha)\} = \{R_{(-1)}(w) : w \in R_{(-n)}(\alpha)\}.$$

Hence,

$$\prod_{w \in \{R_{(-(n+1))}(\alpha)\}} w = \prod_{w \in \{R_{(-n)}(\alpha)\}} \left(\prod_{z \in \{R_{(-1)}(w)\}} z \right) = \prod_{w \in \{R_{(-n)}(\alpha)\}} \left(\frac{-w Q(0)}{P_d} \right),$$

with the second equality following from the $n = 1$ case.

Since $|R_{(-n)}(\alpha)| = d^n$ (not necessarily distinct) this equality becomes

$$\begin{aligned} \prod_{w \in \{R_{(-(n+1))}(\alpha)\}} w &= \left(\frac{-Q(0)}{P_d} \right)^{d^n} \prod_{w \in \{R_{(-n)}(\alpha)\}} w \\ &= \left(\frac{-Q(0)}{P_d} \right)^{d^n} \cdot (\alpha) \left(\frac{-Q(0)}{P_d} \right)^{\left(\frac{d^n - 1}{d - 1} \right)} \\ &= \alpha \left(\frac{-Q(0)}{P_d} \right)^{\left(\frac{d^{n+1} - 1}{d - 1} \right)}, \end{aligned}$$

as desired. \square

The following Theorem is the main result of this paper.

Theorem 3.4. For a given fully symmetric self-similar structure on a finitely ramified fractal K , let V_n denote its sequence of approximating graphs and let P_n denote the probabilistic graph Laplacian of V_n . Arising naturally from the spectral decimation process, there is a rational function $R(z)$, which satisfies the conditions of Lemma 3.3, finite sets $A, B \subset \mathbb{R}$ such that for all $\alpha \in A$, $\beta \in B$, and integers $n, k \geq 0$, there exist functions α_n and β_n^k such that the number of spanning trees of V_n is given by

$$\tau(V_n) = \left| \frac{\left(\prod_{j=1}^{|V_n|} d_j \right)}{\left(\sum_{j=1}^{|V_n|} d_j \right)} \left(\prod_{\alpha \in A} \alpha^{\alpha_n} \right) \left[\prod_{\beta \in B} \left(\beta^{\sum_{k=0}^n \beta_n^k} \left(\frac{-Q(0)}{P_d} \right)^{\sum_{k=0}^n \beta_n^k \left(\frac{d^k - 1}{d - 1} \right)} \right) \right] \right| \quad (3)$$

where d is the degree of $R(z)$, P_d is the leading coefficient of the numerator of $R(z)$, $|V_n|$ is the number of vertices of V_n and d_j is the degree of vertex j in V_n .

Proof of Theorem 3.4. From Kirchhoff's matrix-tree theorem for probabilistic graph Laplacians (Theorem 2.1), we know that

$$\tau(V_n) = \frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} \prod_{j=1}^{|V_n|-1} \lambda_j$$

where λ_j are the non-zero eigenvalues of P_n .

Existence and uniqueness of the rational function $R(z)$ is given Proposition (3.1). After carrying out the inductive calculations using Proposition (3.2) items (1)-(8), we get the sets A and B , and the functions α_n and β_n^k .

To see that the sets A and B are finite. Recall that the functions $R(z)$ and $\phi(z)$ from Proposition (3.2) are rational, thus $R(z)$, $\phi(z)$, and $R(z)\phi(z)$ have finitely many zeroes, poles, and removable singularities. Also, since the matrix D , from writing P_1 in block form to define the Schur Complement, is finite, $\sigma(D)$ is finite. Following items (1)-(8) of Proposition (3.2) these observations imply that A and B are finite sets.

From Proposition (3.2) we know that

$$\{\lambda_j\}_{j=1}^{|V_n|-1} = \bigcup_{\alpha \in A} \{\alpha\} \bigcup_{\beta \in B} \left[\bigcup_{k=0}^n \{R_{-k}(\beta) : \beta_n^k \neq 0\} \right]$$

where the multiplicities of $\alpha \in A$ are given by α_n and the multiplicities of $\{R_{-k}(\beta)\}$ are given by β_n^k . Letting $\lambda_{|V_n|} = 0$.

From items (1)-(8) of Proposition (3.2) it follows that $\forall z \in \{R_{-k}(\beta)\}$ the multiplicity of z depends only on n and k , thus

$$\prod_{j=1}^{|V_n|-1} \lambda_j = \left(\prod_{\alpha \in A} \alpha^{\alpha_n} \right) \left[\prod_{\beta \in B} \left(\prod_{k=0}^n \left(\prod_{z \in \{R_{-k}(\beta)\}} z^{\beta_n^k} \right) \right) \right]$$

From Lemma 4.9 in [32], $R(0) = 0$. From Corollary 1 in [27], it follows that, if we write $R(z) = \frac{P(z)}{Q(z)}$ where $P(z)$ and $Q(z)$ are relatively prime polynomials, then $\deg(P(z)) > \deg(Q(z))$. Thus, the conditions of Lemma 3.3 are satisfied, and applying this theorem gives

$$\begin{aligned} &= \left(\prod_{\alpha \in A} \alpha^{\alpha_n} \right) \left[\prod_{\beta \in B} \left(\prod_{k=0}^n \left(\beta \left(\frac{-Q(0)}{P_d} \right)^{\frac{d^k-1}{d-1}} \right)^{\beta_n^k} \right) \right] \\ &= \left(\prod_{\alpha \in A} \alpha^{\alpha_n} \right) \left[\prod_{\beta \in B} \left(\beta^{\sum_{k=0}^n \beta_n^k} \left(\frac{-Q(0)}{P_d} \right)^{\sum_{k=0}^n \beta_n^k \left(\frac{d^k-1}{d-1} \right)} \right) \right] \end{aligned}$$

Applying Kirchhoff's matrix-tree theorem for probabilistic graph Laplacians (Theorem 2.1), we verify the result. \square

Section 5 of this work will begin with a well known example, the Sierpiński Gasket, and show how to use this theorem to calculate the number of spanning trees on the fractal graphs under consideration. This theorem will then be used to compute the number of spanning trees for three previously unknown examples.

In [13], the authors derived multidimensional polynomial recursion equations to solve explicitly for the number of spanning trees on $SG_d(n)$ with d equal to two, three and four, and on $SG_{d,b}(n)$ with d equal to two and b equal to two and three. They note in that work that it is intriguing that their recursion

relations become more and more complicated as b and d increase, but the solutions remain simple, and comment that with their methods, they do not have a good explanation for this. The following corollary explains why the solutions remain simple.

Corollary 3.5. For a given fully symmetric self-similar structure on a finitely ramified fractal K , with approximating graphs V_n , there exist a finite set of primes $\{p_k\}_{k=1}^r$ and functions $\{f_k : \mathbb{N}_0 \rightarrow \mathbb{N}_0\}_{k=1}^r$ such that

$$\tau(V_n) = \prod_{k=1}^r p_k^{f_k(n)}.$$

Proof of Corollary 3.5. Since $\tau(V_n)$ is a nonnegative integer, and is given by equation (3), the sets A and B are fixed, and self-similarity gives that for any $n \geq 2$ the only prime factors of $\left(\prod_{i=1}^{|V_n|} d_i\right)$ are the prime factors of $\left(\prod_{i=1}^{|V_1|} d_i\right)$. \square

4 Asymptotic Complexity

Let T_n for $n \geq 0$ be a sequence of finite graphs, $|T_n|$ the number of vertices in T_n , and $\tau(T_n)$ denote the number of spanning trees of T_n . $\tau(T_n)$ is called the *complexity* of T_n . The *asymptotic complexity* of the sequence T_n is defined as

$$\lim_{n \rightarrow \infty} \frac{\log(\tau(T_n))}{|T_n|}.$$

When this limit exist, it is called the *asymptotic complexity constant*, or the *tree entropy* of T_n , or the *thermodynamic limit* of T_n .

For any two, finite, connected graphs G_1, G_2 , let $G_1 \vee_{x_1, x_2} G_2$ denote the graph formed by identifying the vertex $x_1 \in G_1$ with vertex $x_2 \in G_2$. Then $\forall x_1 \in G_1, x_2 \in G_2$, it is clear that

$$\tau(G_1 \vee_{x_1, x_2} G_2) = \tau(G_1) \cdot \tau(G_2). \quad (4)$$

Dropping the assumption of full symmetry, we lose the spectral decimation process, but still have the following.

Theorem 4.1. For a given self-similar structure on a finitely ramified fractal K , let V_n denote its sequence of approximating graphs. Let m denote the number of 0-cells of the V_1 graph.

1. If V_1 is a tree, then $\tau(V_n) = 1 \forall n \geq 0$
2. If V_1 is not a tree, then $\log(\tau(V_n)) \in \theta(|V_n|) = \theta(m^n)$

Proof of Theorem 4.1. If V_1 is a tree, then K is a fractal string. Hence $\forall n \geq 0$ V_n is a tree. If V_1 is not a tree, it is m copies of the V_0 graph with vertices identified appropriately. Similarly the V_n graph is m^n copies of the V_0 graph with vertices identified appropriately. Let $V_0 \vee_{x, x}^{m^n} V_0$ denote m^n copies of V_0 each identified to each other at some vertex $x \in V_0$, then clearly for $n \geq 0$

$$\tau(V_n) \geq \tau(V_0 \vee_{x, x}^{m^n} V_0). \quad (5)$$

Since V_1 is not a tree, $|V_0| > 2$, also the V_0 graph is the complete graph on $|V_0|$ vertices, so by Cayley's formula, [45], $\tau(V_0) = |V_0|^{|V_0|-2}$.

Combining this with equation (4) we get that

$$\tau(V_0 \vee_{x,x}^{m^n} V_0) = |V_0|^{|V_0|-2} \cdot m^n$$

and

$$\tau(V_n) \geq |V_0|^{|V_0|-2} \cdot m^n.$$

So for $n \geq 0$,

$$\log(\tau(V_n)) \geq m^n \cdot (|V_0| - 2) \log(|V_0|) \sim |V_n| \quad (6)$$

Since $m^n \sim |V_n|$

Now, V_n can also be constructed by deletion of edges from the graph $K_{|V_n|}$. The deletion-contraction principle, [45], says that for any connected graph G and any edge e in that graph

$$\tau(G) = \tau(G \setminus e) + \tau(G - e),$$

where $G \setminus e$ is the graph formed by contracting e in G and $G - e$ is the graph formed by deleting e from G .

This tells us that deleting edges from graphs decreases the number of spanning trees, thus

$$\tau(V_n) \leq \tau(K_{|V_n|}) = |V_n|^{|V_n|-2}.$$

Since $|V_n| \sim m^n$,

$$\tau(V_n) \lesssim m^{n(m^n-2)},$$

which implies $\forall \epsilon > 0$

$$\lim_{x \rightarrow \infty} \frac{\log(\tau(V_n))}{m^{n(1+\epsilon)}} = 0. \quad (7)$$

Now, suppose that the sequence $\frac{\log(\tau(V_n))}{m^n}$ is unbounded then $\forall M > 0 \exists n_0$ s.t. $\forall n \geq n_0 \frac{\log(\tau(V_n))}{m^n} > M$, but then $\forall \epsilon > 0$ and $\forall n > \frac{n_0}{(1+\epsilon)}$, $\frac{\log(\tau(V_n))}{m^{n(1+\epsilon)}} > M$ which contradicts equation (7). Thus, $\frac{\log(\tau(V_n))}{m^n}$ is bounded and combining this with equation (6) implies $\log(\tau(V_n)) \in \theta(|V_n|)$, as desired. \square

Corollary 4.2. For a given self-similar structure on a finitely ramified fractal K , let V_n denote its sequence of approximating graphs. The following limits exist.

$$\limsup_{n \rightarrow \infty} \frac{\log(\tau(V_n))}{|V_n|}, \quad (8)$$

$$\liminf_{n \rightarrow \infty} \frac{\log(\tau(V_n))}{|V_n|}. \quad (9)$$

We conclude this section with a few conjectures. The first is the natural question to follow from Corollary 4.2. Is additionally requiring full symmetry enough to get convergence?

Conjecture 4.3. For a given fully symmetric self-similar structure on a finitely ramified fractal K , let V_n denote its sequence of approximating graphs. The following limits exist.

$$\lim_{n \rightarrow \infty} \frac{\log(\tau(V_n))}{|V_n|}. \quad (10)$$

The family of fractal trees indexed by the number of branches they possess provide a nice class of examples, and is studied in [20]. These examples show that even though each m -Tree Fractal in the limit is topologically a tree, the number of spanning trees on the approximating graphs grows arbitrarily large. Now, considering the 3-Tree Fractal with graph approximations $V_{3,n}$, using equation (4) it is easy to verify that $\tau(V_{3,n}) = 3^{3^n}$ for $n \geq 0$, and the asymptotic complexity constant is $\frac{\log(3)}{2}$.

Conjecture 4.4. For a given fully symmetric self-similar structure on a finitely ramified fractal K , let V_n denote its sequence of approximating graphs, and c_K denote the asymptotic complexity constant. If V_1 is not a tree, then

$$c_K \geq \frac{\log(3)}{2}.$$

5 Examples

5.1 Sierpiński Gasket

The Sierpiński gasket has been extensively studied (in [39, 3, 28, 34, 4, 18, 22, 37, 40], among others.) It can be constructed as a p.c.f. fractal, in the sense of Kigami [28], in \mathbb{R}^2 using the contractions

$$f_i(x) = \frac{1}{2}(x - q_i) + q_i,$$

for $i = 1, 2, 3$, where the points q_i are the vertices of an equilateral triangle.

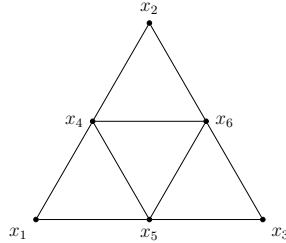


Figure 1: The V_1 network of Sierpiński gasket.

In [13], the following theorem was proven. Here we give a new proof using the method described in Section 3 to show how to use Theorem 3.4.

Theorem 5.1. The number of spanning trees on the Sierpiński gasket at level n is given by

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 5^{h_n}, \quad n \geq 0$$

where

$$f_n = \frac{1}{2}(3^n - 1), g_n = \frac{1}{4}(3^{n+1} + 2n + 1), \text{ and } h_n = \frac{1}{4}(3^n - 2n - 1).$$

Proof of Theorem 5.1. Before applying Theorem 3.4, we make the following observations. It is well known that the V_n network of the Sierpiński gasket has

$$|V_n| = \frac{3^{n+1} + 3}{2} \quad n \geq 0$$

vertices, three of which have degree 2 and the remaining vertices have degree 4. Hence,

$$\frac{\prod_{i=1}^{|V_n|} d_i}{\sum_{i=1}^{|V_n|} d_i} = 2^{3^{n+1}-1} \cdot 3^{-(n+1)}. \quad (11)$$

In [2], they use a result from [3] to carry out spectral decimation for the Sierpiński gasket. In our language, they showed that

$$A = \left\{ \frac{3}{2} \right\}, B = \left\{ \frac{3}{4}, \frac{5}{4} \right\},$$

$$(I) \quad \alpha = \frac{3}{2}, \alpha_n = \frac{3^n+3}{2}, \quad n \geq 0,$$

$$(II) \quad \beta = \frac{3}{4}, \quad n \geq 1$$

$$\beta_n^k = \begin{cases} \frac{3^{n-k-1}+3}{2} & k = 0, \dots, n-1 \\ 0 & k = n, \end{cases}$$

$$(III) \quad \beta = \frac{5}{4}, \quad n \geq 2$$

$$\beta_n^k = \begin{cases} \frac{3^{n-k-1}-1}{2} & k = 0, \dots, n-2 \\ 0 & k = n-1, n \end{cases}$$

and $R(z) = z(5-4z)$. So $d = 2$, $Q(0) = 1$ and $P_d = -4$.

We now use Equation 3 in Theorem 3.4 to calculate $\tau(V_n)$. We have

$$\prod_{\alpha \in A} \alpha^{\alpha_n} = \left(\frac{3}{2} \right)^{\frac{3^n+3}{2}} \quad (12)$$

$$\begin{aligned} \prod_{\beta \in B} \left(\beta^{\sum_{k=0}^n \beta_n^k} \cdot \left(\frac{1}{4} \right)^{\sum_{k=0}^n \beta_n^k (2^k-1)} \right) &= \\ &= \left(\frac{3}{4} \right)^{\sum_{k=0}^{n-1} \left(\frac{3^{n-k-1}+3}{2} \right)} \times \left(\frac{1}{4} \right)^{\sum_{k=0}^{n-1} \left(\frac{3^{n-k-1}+3}{2} \right) (2^k-1)} \\ &\quad \times \left(\frac{5}{4} \right)^{\sum_{k=0}^{n-2} \left(\frac{3^{n-k-1}-1}{2} \right)} \times \left(\frac{1}{4} \right)^{\sum_{k=0}^{n-2} \left(\frac{3^{n-k-1}-1}{2} \right) (2^k-1)} \end{aligned} \quad (13)$$

We sum the expressions in the exponents above.

$$\begin{aligned}\sum_{k=0}^{n-1} \left(\frac{3^{n-k-1} + 3}{2} \right) &= \frac{1}{4} (3^n + 6n - 1) \\ \sum_{k=0}^{n-1} \left(\frac{3^{n-k-1} + 3}{2} \right) (2^k - 1) &= \frac{1}{4} (3^n + 2^{n+2} - 6n - 5) \\ \sum_{k=0}^{n-2} \left(\frac{3^{n-k-1} - 1}{2} \right) &= \frac{1}{4} (3^n - 2n - 1) \\ \sum_{k=0}^{n-2} \left(\frac{3^{n-k-1} - 1}{2} \right) (2^k - 1) &= \frac{1}{4} (3^n - 2^{n+2} + 2n + 3).\end{aligned}$$

All of these equations are valid for $n \geq 2$. Using equations 3, 11, 12, and 13, and simplifying we get:

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 5^{h_n} \quad n \geq 2,$$

as desired. For $n = 1$, equation 11 still holds and the eigenvalues of the probabilistic graph Laplacian are $\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{4}, \frac{3}{4}, 0\}$. So by Theorem 2.1, we get that $\tau(V_1) = 2 \cdot 3^3$. The V_0 network is the complete graph on 3 vertices, thus $\tau(V_0) = 3$. Hence the theorem holds for all $n \geq 0$. \square

As in [13], we immediately have the following Corollary.

Corollary 5.2. The asymptotic growth constant for the Sierpiński Gasket is

$$c = \frac{\log(2)}{3} + \frac{\log(3)}{2} + \frac{\log(5)}{6} \quad (14)$$

5.2 A Non-p.c.f. Analog of the Sierpiński Gasket

As described in [3, 5, 41], this fractal is finitely ramified by not p.c.f. in the sense of Kigami. It can be constructed as a self-affine fractal in \mathbb{R}^2 using 6 affine contractions. One affine contraction has the fixed point $(0, 0)$ and the matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

and the other five affine contractions can be obtained though combining this one with the symmetries of the equilateral triangle on vertices $(0, 0)$, $(1, 0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Figure ?? shows the V_1 network for this fractal.

Theorem 5.3. The number of spanning trees on the non-p.c.f. analog of the Sierpiński gasket at level n is given by

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 5^{h_n}, \quad n \geq 0$$

where

$$\begin{aligned}f_n &= \frac{2}{25} (11 \cdot 6^n - 30n - 11), \quad g_n = \frac{1}{5} (2 \cdot 6^n + 3), \text{ and} \\ h_n &= \frac{1}{25} (4 \cdot 6^n + 30n - 4).\end{aligned}$$

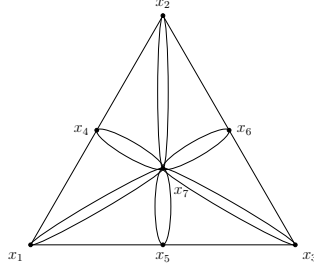


Figure 2: The V_1 network of the non-p.c.f. analog of the Sierpiński gasket.

Before the proof, we need a few results.

Lemma 5.4. The V_n network of the non-p.c.f. analog of the Sierpiński gasket, for $n \geq 0$, has

$$\frac{4 \cdot 6^n + 11}{5}$$

vertices. Among these vertices,

- (i) 3 have degree 2^{n+1} ,
- (ii) 6^{k-1} have degree $3 \cdot 2^{n-k+2}$ for $1 \leq k \leq n$, and
- (iii) $3 \cdot 6^{k-1}$ have degree 2^{n-k+2} for $1 \leq k \leq n$.

Proof of Lemma 5.4. We first describe how the V_n network is constructed, then prove the Lemma.

For $n = 0$, V_0 is the complete graph on vertices $\{x_1, x_2, x_3\}$, one triangle (the V_0 network) and 3 corners of degree 2 $\{x_1, x_2, x_3\}$ are born at level 0.

For $n = 1$, from the triangle born on level 0, 6 triangles are born. For example one of these triangles is the complete graph on $\{x_2, x_4, x_7\}$. 3 corners of degree 4 are born, they are $\{x_4, x_5, x_6\}$ and one center is born $\{x_7\}$ of degree 12.

For $n \geq 2$, from each triangle born at level $n - 1$, 6 triangles are born, 3 corners of degree 4 are born and 1 center of degree 12 is born. Each corner born at level $n - 1$ gains 4 edges. Each center born at level $n - 1$ gains 12 edges. Each corner born at level $n - 2$ gains $2 \cdot 4$ edges. Each center born at level $n - 2$ gains $2 \cdot 12$ edges. In general, for $1 \leq k \leq n - 1$, each corner born at level $n - k$ gains $2^{k-1} \cdot 4$ edges, and each center born at level $n - k$ gains $2^{k-1} \cdot 12$ edges. The corners born at level 0 gain 2^n edges.

From this construction we see that, for $n \geq 0$ the V_n network has

$$3 + 4 \cdot \sum_{j=0}^{n-1} 6^j = \frac{4 \cdot 6^n + 11}{5}$$

vertices, as desired.

On the V_n network, for $n \geq 0$, the 3 corners born on level 0 have degree

$$2 + \sum_{j=1}^n 2^j = 2^{n+1},$$

which verifies item (i).

Following the construction, we see that on the V_n network, for $n \geq 1$, there are 6^{n-1} centers born at level n , each with degree 12. There are 6^{n-2} centers born at level $n-1$, each with degree $12+12$. In general, for $0 \leq k \leq n$, there are 6^{n-k-1} centers born at level $n-k$, each with degree

$$12 + 12 \cdot \sum_{j=0}^{k-1} 2^j = 3 \cdot 2^{k+2}.$$

After changing indices, item (ii) follows, noting that item (ii) is a vacuous statement for $n = 0$.

Similarly, for $0 \leq k \leq n$, in the V_n network, there are $3 \cdot 6^{n-k-1}$ corners born at level $n-k$. Each of which have degree

$$4 + 4 \cdot \sum_{j=0}^{k-1} 2^j = 2^{k+2}.$$

After changing indices, item (iii) follows, noting that item (iii) is a vacuous statement for $n = 0$. \square

Corollary 5.5. For the V_n network of the non-p.c.f. analog of the Sierpiński gasket, for $n \geq 1$, we have

$$\frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} = 2^{\frac{1}{25}(44 \cdot 6^n + 30n + 6)} \cdot 3^{\frac{1}{5}(6^n - 5n - 6)}.$$

We are now ready for the proof of the main theorem in this section.

Proof of Theorem 5.3. We apply Theorem 3.4. In [3], they use a result from [2] to carry out spectral decimation for the non-p.c.f. analog of the Sierpiński gasket. In our language, they showed that

$$A = \left\{ \frac{3}{2} \right\}, \text{ and } B = \left\{ \frac{3}{4}, \frac{5}{4}, \frac{1}{2}, 1 \right\}.$$

Rephrasing their results in our language, for $n \geq 2$ the following hold:

$$(I) \quad \alpha = \frac{3}{2}, \quad \alpha_n = 6^{n-1} + 1,$$

$$(II) \quad \beta = \frac{3}{4},$$

$$\beta_n^k = \begin{cases} 6^{n-k-2} + 1 & k = 0, \dots, n-2 \\ 2 & k = n-1 \\ 0 & k = n, \end{cases}$$

$$(III) \quad \beta = \frac{5}{4},$$

$$\beta_n^k = \begin{cases} 6^{n-k-2} + 1 & k = 0, \dots, n-2 \\ 2 & k = n-1 \\ 0 & k = n, \end{cases}$$

(IV) $\beta = \frac{1}{2}$,

$$\beta_n^k = \begin{cases} \frac{11 \cdot 6^{n-k-2} - 6}{5} & k = 0, \dots, n-2 \\ 0 & k = n-1, n, \end{cases}$$

(V) $\beta = 1$,

$$\beta_n^k = \begin{cases} \frac{6^{n-k} - 6}{5} & k = 0, \dots, n-2 \\ 0 & k = n-1, n, \end{cases}$$

and

$$R(z) = \frac{-24z(z-1)(2z-3)}{14z-15}.$$

So $d = 3$, $Q(0) = -15$ and $P_d = -48$.

We now use Equation 3 in Theorem 3.4 to calculate $\tau(V_n)$. We have from (I),

$$\prod_{\alpha \in A} \alpha^{\alpha_n} = \left(\frac{3}{2}\right)^{6^{n-1}+1}. \quad (15)$$

From (II),(III),(IV), and (V), and to calculate

$$\prod_{\beta \in B} \left(\beta^{\sum_{k=0}^n \beta_n^k} \cdot \left(\frac{15}{48}\right)^{\sum_{k=0}^n \beta_n^k \left(\frac{d^k-1}{d-1}\right)} \right), \quad (16)$$

the relevant summations are,

$$\begin{aligned} & \left[\sum_{k=0}^{n-2} (6^{n-k-2} + 1) \right] + 2 = \frac{1}{5} (6^{n-1} + 5n + 4), \\ & \left[\sum_{k=0}^{n-2} (6^{n-k-2} + 1) \left(\frac{3^k-1}{2}\right) \right] + (3^{n-1} - 1) = \frac{1}{60} (4 \cdot 6^{n-1} + 65 \cdot 3^{n-1} - 30n - 39), \\ & \sum_{k=0}^{n-2} \frac{11 \cdot 6^{n-k-2} - 6}{5} = \frac{1}{25} (11 \cdot 6^{n-1} - 30n + 19), \\ & \sum_{k=0}^{n-2} \left(\frac{11 \cdot 6^{n-k-2} - 6}{5} \right) \left(\frac{3^k-1}{2}\right) = \frac{1}{25} (22 \cdot 6^{n-2} - 50 \cdot 3^{n-2} + 15n - 2), \text{ and} \\ & \sum_{k=0}^{n-2} \left(\frac{6^{n-k} - 6}{5} \right) \left(\frac{3^k-1}{2}\right) = \frac{1}{50} (4 \cdot 6^n - 25 \cdot 3^n + 30n + 21). \end{aligned}$$

All of these equations are valid for $n \geq 2$ and combining with Corollary 5.5, we see that

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 5^{h_n}, \quad n \geq 2$$

where f_n , g_n , and h_n are as claimed. For $n = 0$, since the V_0 graph is the complete graph on three vertices, $\tau(V_0) = 3$ by Cayley's Formula, as desired. For $n = 1$, from [3] the eigenvalues of P_1 are $\{\frac{5}{4}, \frac{5}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{4}, \frac{3}{4}, 0\}$ and using Corollary 5.5 for $n = 1$, we apply Theorem 2.1 to see that $\tau(V_1) = 2^2 \cdot 3^3 \cdot 5^2$, as desired. \square

Corollary 5.6. The asymptotic growth constant for the non-p.c.f. analog of the Sierpiński Gasket is

$$c = \frac{11 \cdot \log(2)}{10} + \frac{\log(3)}{2} + \frac{\log(5)}{5} \quad (17)$$

5.3 Diamond Fractal

The diamond self-similar hierarchical lattice appeared as an example in several physics works, including [24], [25], and [23]. In [2], the authors modify the standard results for the unit interval $[0, 1]$ to develop the spectral decimation method for this fractal, hence Theorem 3.4 still applies. Figure ?? shows the V_1 and V_2 networks for this.

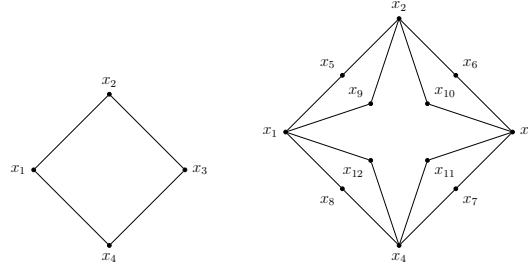


Figure 3: The V_1 and V_2 network of the Diamond fractal.

Theorem 5.7. The number of spanning trees on the Diamond fractal at level n is given by

$$\tau(V_n) = 2^{\frac{2}{3}(4^n - 1)} \quad n \geq 1.$$

Before we begin the proof, we need a few results.

Lemma 5.8. The V_n network of the Diamond fractal, for $n \geq 1$, has

$$\frac{(4 + 2 \cdot 4^n)}{3}$$

vertices. Among these vertices,

- (i) $2 \cdot 4^{n-k}$ have degree 2^k for $1 \leq k \leq n-1$
- (ii) 4 have degree 2^n .

Remark 5.9. In [2], the number of vertices of V_n is incorrect as stated in Theorem 7.1(ii). We correct this here and provide a proof.

Proof of Lemma 5.8. We first describe how the V_n network is constructed, then prove the Lemma. When $n = 1$, V_1 has four vertices of degree 2 and one diamond, this diamond is the graph of V_1 . We say these vertices and diamond are born at level 1.

When $n = 2$, from the diamond born on level 1, 4 diamonds are born. We say these diamonds are born on level 2. For each of the diamonds born on

level 2, 2 vertices of degree 2 are born. We say these vertices are born on level 2. Using the notation $G = \langle V, E \rangle$ where G is the graph, V is the graph's vertex set and E is the graph's edge set. An example diamond born at level 2 is $\langle V, E \rangle$, where

$$V = \{x_1, x_5, x_2, x_9\}$$

$$E = \{x_1x_5, x_5x_2, x_2x_9, x_9x_1\}$$

which gives birth to x_5 and x_9 . Every vertex born on level 1 gains 2 more edges.

For $n \geq 2$, from each diamond born on level $n - 1$, 4 diamonds are born at level n . For each of the diamonds born on level n , 2 vertices of degree 2 are born at level n . Every vertex born on level $n - 1$, gains 2 more edges. Every vertex born on level $n - 2$, gains 2^2 more edges. In general, every vertex born on level $n - k$, gains 2^k more edges for $1 \leq k \leq n - 1$.

From this construction, we see that at level n , for $n \geq 1$, there are 4^{k-1} diamonds born at level k , $1 \leq k \leq n$, $2 \cdot 4^{k-1}$ vertices born at level k , $2 \leq k \leq n$ and 4 vertices born at level 1. Thus, the V_n network has

$$4 + \sum_{k=2}^n 2 \cdot 4^{k-1} = \frac{(4 + 2 \cdot 4^n)}{3} \text{ vertices, as desired.}$$

In the V_n network, the 4 vertices born at level 1 have degree

$$2 + \sum_{j=1}^{n-1} 2^j = 2^n,$$

which verifies item (ii) of the Proposition.

In the V_n network, the $2 \cdot 4^{k-1}$ vertices born on level k , $2 \leq k \leq n$, have degree

$$2 + \sum_{j=1}^{n-k} 2^j = 2^{n-k+1}.$$

changing indices, this verifies item (i) of the Lemma. □

Corollary 5.10. For the V_n network of the Diamond fractal, we have

$$\frac{\prod_{i=1}^{|V_n|} d_i}{\sum_{i=1}^{|V_n|} d_i} = 2^{\frac{1}{9}(2 \cdot 4^{n+1} - 6n - 17)}. \quad (18)$$

We now return to a proof the the main theorem of this section.

Proof of Theorem 5.7. We apply Theorem 3.4. In [2], they carry out spectral decimation for the Diamond fractal. In our language, they showed that

$$A = \{2\}, \text{ and } B = \{1\}.$$

For $n \geq 1$, the following hold:

(I) $\alpha = 2, \alpha_n = 1$

(II) $\beta = 1,$

$$\beta_n^k = \begin{cases} \frac{4^{n-k}+2}{3} & k = 0, \dots, n-1 \\ 0 & k = n, \end{cases}$$

and

$$R(z) = 2z(2-z).$$

So $d = 2$, $Q(0) = 1$, and $P_d = -2$. We now use Equation 3 in Theorem 3.4 to calculate $\tau(V_n)$.

$$\prod_{\alpha \in A} \alpha^{\alpha_n} = 2^1 \quad (19)$$

$$\prod_{\beta \in B} \left(\beta^{\sum_{k=0}^n \beta_n^k} \cdot \left(\frac{1}{2} \right)^{\sum_{k=0}^n \beta_n^k (2^k - 1)} \right) = 2^{-\frac{1}{9}(2 \cdot 4^n - 6n - 2)} \quad (20)$$

the relevant summation is,

$$\sum_{k=0}^{n-1} \left(\frac{4^{n-k} + 2}{3} \right) (2^k - 1) = \frac{1}{9} (2 \cdot 4^n - 6n - 2).$$

Combining this with Corollary 5.10, we have that

$$\tau(V_n) = 2^{\frac{2}{3}(4^n - 1)} \quad n \geq 1$$

as desired. \square

Corollary 5.11. The asymptotic growth constant for the Diamond fractal is

$$c = \log(2) \quad (21)$$

5.4 Hexagasket

The hexagasket, is also known as the hexakun, a polygasket, a 6-gasket, or a $(2, 2, 2)$ -gasket, see [3, 28, 1, 9, 39, 41, 46, 47]. The V_1 network of the hexagasket is shown in the figure below.

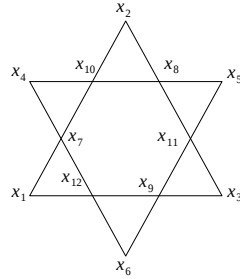


Figure 4: The V_1 network of the Hexagasket.

Theorem 5.12. The number of spanning trees on the Hexagasket at level n is given by

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 7^{h_n} \quad n \geq 0.$$

where

$$\begin{aligned} f_n &= \frac{1}{225} (27 \cdot 6^{n+1} - 100 \cdot 4^n - 60n - 62) \\ g_n &= \frac{1}{25} (4 \cdot 6^{n+1} + 5n + 1) \\ h_n &= \frac{1}{25} (6^n - 5n - 1). \end{aligned}$$

Proof of Theorem 5.12. We apply Theorem 3.4. From [3] it is known that

$$|V_n| = \frac{(6 + 9 \cdot 6^n)}{5} \quad n \geq 0,$$

of these vertices, $\frac{6(6^n-1)}{5}$ have degree 4, and the remaining $\frac{(12+3 \cdot 6^n)}{5}$ have degree 2. So we compute

$$\frac{\prod_{j=1}^{|V_n|} d_j}{\sum_{j=1}^{|V_n|} d_j} = 2^{(3 \cdot 6^n - n - 1)} \cdot 3^{-(n+1)} \quad (22)$$

for $n \geq 0$.

In [3], they use a result from [2] to carry out spectral decimation for the Hexagasket. We note that in [3] Theorem 6.1 (v) and (vi), the bounds on k should be $0 \leq k \leq n-1$ and in (vii) the bounds should be $0 \leq k \leq n-2$. This can be verified using Table 2 in the same paper. In our language they showed that

$$A = \left\{ \frac{3}{2} \right\}, \text{ and } B = \left\{ 1, \frac{1}{4}, \frac{3}{4}, \frac{3+\sqrt{2}}{4}, \frac{3-\sqrt{2}}{4} \right\},$$

and for $n \geq 2$ the following hold:

$$(I) \quad \alpha = \frac{3}{2}, \quad \alpha_n = \frac{(6+4 \cdot 6^n)}{5},$$

$$(II) \quad \beta = 1,$$

$$\beta_n^k = \begin{cases} 1 & k = 0, \dots, n-1 \\ 0 & k = n, \end{cases}$$

$$(III) \quad \beta = \frac{1}{4}, \frac{3}{4},$$

$$\beta_n^k = \begin{cases} \frac{(6+4 \cdot 6^{n-k-1})}{5} & k = 0, \dots, n-1 \\ 0 & k = n, \end{cases}$$

$$(IV) \quad \beta = \frac{3+\sqrt{2}}{4}, \frac{3-\sqrt{2}}{4},$$

$$\beta_n^k = \begin{cases} \frac{(6^{n-k-1}-1)}{5} & k = 0, \dots, n-2 \\ 0 & k = n-1, n, \end{cases}$$

$$R(z) = \frac{2z(z-1)(7-24z+16z^2)}{(2z-1)}.$$

So $d = 4$, $Q(0) = -1$ and $P_d = 32$.

We now use equation 3 in Theorem 3.4 to calculate $\tau(V_n)$. The relevant sums are

$$\sum_{k=0}^{n-1} \frac{(4^k - 1)}{3} = \frac{(4^n - 3n - 1)}{9} \quad (23)$$

$$\sum_{k=0}^{n-1} \frac{(6 + 4 \cdot 6^{n-k-1})}{5} = \frac{2 \cdot (2 \cdot 6^n + 15n - 2)}{25} \quad (24)$$

$$\sum_{k=0}^{n-1} \frac{(6 + 4 \cdot 6^{n-k-1})}{5} \frac{(4^k - 1)}{3} = \frac{(6^{n+1} - 30n - 6)}{75} \quad (25)$$

$$\sum_{k=0}^{n-2} \frac{(6^{n-k-1} - 1)}{5} = \frac{(6^n - 5n - 1)}{25} \quad (26)$$

$$\sum_{k=0}^{n-2} \frac{(6^{n-k-1} - 1)}{5} \frac{(4^k - 1)}{3} = \frac{(9 \cdot 6^n - 25 \cdot 4^n + 30n + 16)}{450} \quad (27)$$

Combining these using equations 3 and 22, after simplifying we get

$$\tau(V_n) = 2^{f_n} \cdot 3^{g_n} \cdot 7^{h_n} \quad n \geq 2.$$

Where f_n, g_n , and h_n are as claimed.

For $n=1$, equation 22 still holds and from [3] we know the eigenvalues of the probabilistic graph Laplacian on V_1 are $\{1, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 0\}$. So by Theorem 2.1, we get that $\tau(V_1) = 2^2 \cdot 3^6$, thus the theorem holds for $n = 1$. The V_0 network is the complete graph on 3 vertices, thus $\tau(V_0) = 3$. Hence the theorem holds for all $n \geq 0$. \square

Corollary 5.13. The asymptotic growth constant for the Hexagasket is

$$c = \frac{2 \cdot \log(2)}{5} + \frac{8 \cdot \log(3)}{15} + \frac{\log(7)}{45} \quad (28)$$

References

- [1] Bryant Adams, S. Alex Smith, Robert S. Strichartz, and Alexander Teplyaev. The spectrum of the Laplacian on the pentagasket. In *Fractals in Graz 2001*, Trends Math., pages 1–24. Birkhäuser, Basel, 2003.
- [2] N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst, and A. Teplyaev. Vibration modes of $3n$ -gaskets and other fractals. *J. Phys. A*, 41(1):015101, 21, 2008.
- [3] N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst, and A. Teplyaev. Vibration spectra of finitely ramified, symmetric fractals. *Fractals*, 16(3):243–258, 2008.

- [4] J. Bellissard. Renormalization group analysis and quasicrystals. In *Ideas and methods in quantum and statistical physics (Oslo, 1988)*, pages 118–148. Cambridge Univ. Press, Cambridge, 1992.
- [5] Oren Ben-Bassat, Robert S. Strichartz, and Alexander Teplyaev. What is not in the domain of the Laplacian on Sierpinski gasket type fractals. *J. Funct. Anal.*, 166(2):197–217, 1999.
- [6] Norman Biggs. *Algebraic graph theory*. Cambridge University Press, London, 1974. Cambridge Tracts in Mathematics, No. 67.
- [7] Béla Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.
- [8] J. A. Bondy and U. S. R. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics*. Springer, New York, 2008.
- [9] Brighid Boyle, Kristin Cekala, David Ferrone, Neil Rifkin, and Alexander Teplyaev. Electrical resistance of N -gasket fractal networks. *Pacific J. Math.*, 233(1):15–40, 2007.
- [10] T.J.N. Brown, R.B. Mallion, P. Pollak, and A. Roth. Some methods for counting the spanning trees in labelled molecular graphs, examined in relation to certain fullerenes. *Discrete Applied Mathematics*, 67(1–3):51 – 66, 1996. Chemistry and Discrete Mathematics.
- [11] Robert Burton and Robin Pemantle. Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances. *Ann. Probab.*, 21(3):1329–1371, 1993.
- [12] Seth Chaiken. A combinatorial proof of the all minors matrix tree theorem. *SIAM J. Algebraic Discrete Methods*, 3(3):319–329, 1982.
- [13] Shu-Chiuan Chang, Lung-Chi Chen, and Wei-Shih Yang. Spanning trees on the Sierpinski gasket. *J. Stat. Phys.*, 126(3):649–667, 2007.
- [14] Shu-Chiuan Chang and Robert Shrock. Some exact results for spanning trees on lattices. *J. Phys. A*, 39(20):5653–5658, 2006.
- [15] Shu-Chiuan Chang and Wenya Wang. Spanning trees on lattices and integral identities. *J. Phys. A*, 39(33):10263–10275, 2006.
- [16] Charles J. Colbourn. *The combinatorics of network reliability*. International Series of Monographs on Computer Science. The Clarendon Press Oxford University Press, New York, 1987.
- [17] Bruce J. Collings. Characteristic polynomials by diagonal expansion. *The American Statistician*, 37(3), Aug. 1983.
- [18] Kyallee Dalrymple, Robert S. Strichartz, and Jade P. Vinson. Fractal differential equations on the Sierpinski gasket. *J. Fourier Anal. Appl.*, 5(2-3):203–284, 1999.
- [19] Deepak Dhar. Theoretical studies of self-organized criticality. *Phys. A*, 369(1):29–70, 2006.

- [20] Daniel J. Ford and Benjamin Steinhurst. Vibration spectra of the m -tree fractal. *Fractals*, 18(2):157–169, 2010.
- [21] C. M. Fortuin and P. W. Kasteleyn. On the random-cluster model. I. Introduction and relation to other models. *Physica*, 57:536–564, 1972.
- [22] M. Fukushima and T. Shima. On a spectral analysis for the Sierpiński gasket. *Potential Anal.*, 1(1):1–35, 1992.
- [23] Yuval Gefen, Amnon Aharony, and Benoit B. Mandelbrot. Phase transitions on fractals. III. Infinitely ramified lattices. *J. Phys. A*, 17(6):1277–1289, 1984.
- [24] Yuval Gefen, Amnon Aharony, and Benoit B. Mandelbrot. Phase transitions on fractals. I. Quasilinear lattices. *J. Phys. A*, 16(6):1267–1278, 1983.
- [25] Yuval Gefen, Amnon Aharony, Yonathan Shapir, and Benoit B. Mandelbrot. Phase transitions on fractals. II. Sierpiński gaskets. *J. Phys. A*, 17(2):435–444, 1984.
- [26] Frank Harary and Edgar M. Palmer. *Graphical enumeration*. Academic Press, New York, 1973.
- [27] Katheryn Hare, Benjamin Steinhurst, Alexander Teplyaev, and Denglin Zhou. Disconnected julia sets and gaps in the spectrum of laplacians on symmetric finitely ramified fractals. *arXiv:1105.1747v2*, 2011.
- [28] Jun Kigami. *Analysis on fractals*, volume 143 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.
- [29] G. Kirchhoff. Über die auflösung der gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer ströme geführt wird. *Ann. Phys. Chem.*, 72:497–508, 1847.
- [30] Russell Lyons. Asymptotic enumeration of spanning trees. *Combin. Probab. Comput.*, 14(4):491–522, 2005.
- [31] Russell Lyons. Identities and inequalities for tree entropy. *Combin. Probab. Comput.*, 19(2):303–313, 2010.
- [32] Leonid Malozemov and Alexander Teplyaev. Self-similarity, operators and dynamics. *Math. Phys. Anal. Geom.*, 6(3):201–218, 2003.
- [33] Volker Metz. The short-cut test. *J. Funct. Anal.*, 220(1):118–156, 2005.
- [34] R. Rammal. Spectrum of harmonic excitations on fractals. *J. Physique*, 45(2):191–206, 1984.
- [35] Fritz Runge and Horst Sachs. Berechnung der Anzahl der Gerüste von Graphen und Hypergraphen mittels deren Spektren. *Math. Balkanica*, 4:529–536, 1974. Papers presented at the Fifth Balkan Mathematical Congress (Belgrade, 1974).

- [36] T. Shima. The eigenvalue problem for the Laplacian on the Sierpiński gasket. In *Asymptotic problems in probability theory: stochastic models and diffusions on fractals* (Sanda/Kyoto, 1990), volume 283 of *Pitman Res. Notes Math. Ser.*, pages 279–288. Longman Sci. Tech., Harlow, 1993.
- [37] Tadashi Shima. On eigenvalue problems for Laplacians on p.c.f. self-similar sets. *Japan J. Indust. Appl. Math.*, 13(1):1–23, 1996.
- [38] Robert Shrock and F. Y. Wu. Spanning trees on graphs and lattices in d dimensions. *J. Phys. A*, 33(21):3881–3902, 2000.
- [39] Robert S. Strichartz. *Differential equations on fractals*. Princeton University Press, Princeton, NJ, 2006. A tutorial.
- [40] Alexander Teplyaev. Spectral analysis on infinite Sierpiński gaskets. *J. Funct. Anal.*, 159(2):537–567, 1998.
- [41] Alexander Teplyaev. Harmonic coordinates on fractals with finitely ramified cell structure. *Canad. J. Math.*, 60(2):457–480, 2008.
- [42] Elmar Teufl and Stephan Wagner. The number of spanning trees of finite sierpiński graphs. In: *Fourth Colloquium on Mathematics and Computer Science*, Nancy:411–414, 2006.
- [43] Elmar Teufl and Stephan Wagner. Enumeration problems for classes of self-similar graphs. *J. Combin. Theory Ser. A*, 114(7):1254–1277, 2007.
- [44] Elmar Teufl and Stephan Wagner. The number of spanning trees in self-similar graphs. *Ann. Comb.*, 15(2):355–380, 2011.
- [45] W. T. Tutte. *Graph theory*, volume 21 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001. With a foreword by Crispin St. J. A. Nash-Williams, Reprint of the 1984 original.
- [46] Jeremy T. Tyson and Jang-Mei Wu. Characterizations of snowflake metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 30(2):313–336, 2005.
- [47] Jeremy T. Tyson and Jang-Mei Wu. Quasiconformal dimensions of self-similar fractals. *Rev. Mat. Iberoam.*, 22(1):205–258, 2006.
- [48] W.-J. Tzeng and F. Y. Wu. Spanning trees on hypercubic lattices and nonorientable surfaces. *Appl. Math. Lett.*, 13(7):19–25, 2000.
- [49] D. J. A. Welsh. *Complexity: knots, colourings and counting*, volume 186 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.
- [50] F. Y. Wu. Number of spanning trees on a lattice. *J. Phys. A*, 10(6):L113–L115, 1977.
- [51] F. Y. Wu. The Potts model. *Rev. Modern Phys.*, 54(1):235–268, 1982.
- [52] Robert M. Ziff. Generalized cell–dual-cell transformation and exact thresholds for percolation. *Phys. Rev. E* (3), 73(1):016134, 6, 2006.